EEC S 127 Final Cheatsheet Jeffrey Shen

② Linear Algebra

Independence: $\sum_{i=1}^{m} \lambda_i x_i = 0 \quad \lambda \in \mathbb{R}^m \quad \lambda = 0,$
no vector as linear combination of others

Scalar Product: $x^T y = \sum_{i=1}^{n} x_i y_i$ also $\langle x, y \rangle$

3 Popular Norms

1. L2 Norm (Euclidean): $\|x\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x^T x}$
2. L1 Norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$
3. L∞ Norm: $\|x\|_\infty := \max_{1 \le i \le n} |x_i|$

Frobenius Norm: $\|A\|_F := \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2} = \sqrt{Tr(A^T A)}$

Orthonormal: $u_i^T u_j = 0,\ i \ne j$ and $\|u_i\|_2 = 1$

affine $\tilde{f}: \mathbb{R}^n \to \mathbb{R} \quad \tilde{f}(x) = f(x) - f(0)$ is linear
can be expresse v scalar product $f(x) = a^T x + b$

Trace: $Tr A = \sum_{i=1}^{n} A_{ii}$ sum of diagonal elements

Diagonal Matrices: $diag(a)$: $A_{ij} = 0$ when $i \ne j$

Fundamental Theorem of Linear Algebra

Range of a matrix is the orthogonal complement of the nullspace of its transpose

$$R(A)^\perp = N(A^T)$$

Ordinary Least Squares (OLS / LS): $\min_x \|Ax - y\|_2^2$

$$x^* = (A^T A)^{-1} A^T y$$

$Tr(AB) = Tr(BA)$

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

Eigenvalues 2x2

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \underbrace{m}_{\frac{a+d}{2}} \pm \sqrt{m^2 - \underbrace{p}_{\det(M) = ad-bc}}$

Optimization Problems

$LP \subseteq QP \subseteq QCQP \subseteq SOCP \subseteq CP$

Linear Programming (LP)
obj: linear
constraints: affine inequality
$\min c^T x : a_i^T x \le b_i$

Quadratic Programming (QP)
obj: quadratic convex, Q is PSD
constraint: affine
$\min_x c^T x + x^T Q x : Ax \le b$

Quadratically Constrained Quadratic Problem (QCQP)
obj: quadratic, all Q PSD
constraint: quadratic

$$\min_x a_0^T x + x^T Q_0 x : x^T Q_i x + a_i^T x \le b_i$$

Second-Order Cone Program (SOCP)
obj: convex linear
constraint: require affine function to lie second-order cone in $\mathbb{R}^{n+1}$

$\min_x c^T x$: Inequality: $\|A_i x + b_i\|_2 \le c_i^T x + d_i$
conic: $(A_i x + b_i, c_i^T x + d_i)$

Conic Optimization $\min c^T x$
$Ax \le b$
$Cx = d$

Spectral Theorem (Symmetric eigenvalue decomposition (SED))

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T = U \Lambda U^T, \quad \Lambda = diag(\lambda_1, \dots, \lambda_n)$$

$$Tr\, \Sigma = \frac{1}{m} X X^T = \frac{1}{m} \|X\|_F^2$$

Total Variance: $Tr\, \Sigma = Tr(U \Lambda U^T) = Tr(U^T U \Lambda) = Tr \Lambda = \lambda_1 + \dots + \lambda_n$

Ratio of variance explained: $\frac{\lambda_1 + \lambda_2}{\lambda_1 + \dots + \lambda_n}$

Invertibility: $I - A$ invertible when A does not have eigenvalue of 1

Cauchy-Schwartz Inequality: $|z^T y| \le \|y\|_2 \cdot \|z\|_2$

Singular Value Decomposition

Decomposition of form

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T = U \tilde{S} V^T, \quad \tilde{S} := \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$$

$r \le \min(m, n)$
$U \in \mathbb{R}^{m \times m}$
$V \in \mathbb{R}^{n \times n}$ orthogonal matrices
$S := diag(\sigma_1 \dots \sigma_r)$

Optimal Set of the OLS problem

$p^* := \min_x \|Ax - y\|_2$ expressed as $\underbrace{x^{opt}}_{\text{min norm point}} = A^+ y + N(A)$

$x^* = A^+ y = (A^T A)^{-1} A^T y$

Positive Semi Definite (PSD): $A \succeq 0$ iff associated quadratic form q non-negative everywhere

$$q(x) \ge 0 \ \forall x \in \mathbb{R}^n$$

PSD $Q = PDP^T$
$PD^{1/2} D^{1/2} P^T$
$= UU^T$

$A = A^{1/2} A^{1/2}$
$= (PD^{1/2} P^T)(PDP^T) = PDP^T$

Convexity

Find convex / concave — convex: x^2 ∪, concave: $\sqrt{x}$ ⌒

1) find points, and check midpoint
2) second derivative test
3) function definitions

- domain must be convex
- $M \to \|M\|_2$ norm convex
- Affine variable transformations preserve convexity
- gives global information
- if twice differentiable, f convex iff Hessian matrix $\nabla^2 f$ is PSD everywhere
- function convex if pointwise max of linear functions

Pointwise Maximum: if $(f_\alpha)_{\alpha \in A}$, function $f(x) := \max_{\alpha \in A} f_\alpha(x)$

③ Convex Models

Linear Programs: optimization problem w/ linear objective and affine inequality constraints. $\min_x f_0(x) : f_i(x) \le 0,\ i = 1 \dots m$

Quadratic Program: optimization problem where
1) constraint functions are affine 2) objective function f_0 quadratic convex

$f_0(x) = c^T x + x^T Q x$: vector $c \in \mathbb{R}^n$, $Q = Q^T \succeq 0$

Rotated Second-order cone: $K_p = \{(x, y, z) \in \mathbb{R}^{p+2} : x^T x \le yz,\ y \ge 0,\ z \ge 0\}$

Second-order cone inequality: $\|Ax + b\|_2 \le c^T x + d$ — $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

Standard forms

Inequality form: $\min_x c^T x : \|A_i x + b_i\|_2 \le c_i^T x + d,\ i = 1, \dots m$

Conic form: $\min_x c^T x : (A_i x + b_i, c_i^T x + d_i) \in K_p,\ i = 1 \dots m$

Quadratically constrained quadratic programming:

$$\min_x a_0^T x + x^T Q_0 x : x^T Q_i x + a_i^T x \le b_i,\ i = 1 \dots m$$

Duality

Primal dual: primal $\min_x c^T x : Px \ge b,\ x \ge 0$

dual: $\max_\lambda \lambda^T b : c \ge P^T \lambda,\ \lambda \ge 0$

$$L(x, \lambda, \Delta) = c^T x + \lambda^T (b - Px) - \Delta^T x = (c - P^T \lambda - \Delta)^T x + \lambda^T b$$

$$g(\lambda, \Delta) = \begin{cases} \lambda^T b & \text{if } c = P^T \lambda + \Delta \\ -\infty & \text{otherwise} \end{cases}$$

$p^* = \min_x c^T x,\ Ax \le b$

$$L(x, \lambda) = c^T x + \lambda^T (Ax - b) = (c + A^T \lambda)^T x - \lambda^T b$$

$$g(\lambda) = \begin{cases} -\infty & \text{if } c + A^T \lambda \ne 0 \\ -\lambda^T b & \text{if } c + A^T \lambda = 0 \end{cases}$$

max $g(\lambda)$ over $\lambda \ge 0$

$d^* = \max_\lambda -\lambda^T b : c + A^T \lambda = 0,\ \lambda \ge 0$

weak duality: non convex minimization: "primal" problem
- find lower bound on optimal value of primal

dual: lower bound value of optimization problem, convex problem

Strong Duality: holds if $p^* = d^*$, weak $p^* \ge d^*$

Slater's Sufficient condition for strong duality: if
- primal problem is convex
- strictly feasible, $x_0 \in \mathbb{R}^n$ st $Ax_0 = b,\ f_i(x_0) < 0,\ i = 1 \dots m$

then strong duality holds

Convex Constrained Optimization Problem in standard form:

$p^* := \min_x f_0(x) : Ax = b,\ f_i(x) \le 0,\ i = 1 \dots m$

Lagrangian: $L(x, \lambda, v) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + v^T (Ax - b)$

Dual function: $g(\lambda, v) = \min_x f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + v^T (Ax - b)$

$d^* = \max_{\lambda \ge 0, v} g(\lambda, v)$

Sion's minimax

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y)$$

Lagrange function: $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ depends on primal variables and additional variable $y \in \mathbb{R}^m$: **dual variable**

$$L(x, y) = f_0(x) + \sum_{i=1}^m y_i f_i(x) = f_0(x) + y^T f(x)$$

Minimax Inequality: for any function two var $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

$$\min_{x \in X} \max_{y \in Y} L(x, y) \ge \max_{y \in Y} \min_{x \in X} L(x, y) \quad X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$$

Dual Function: For $\forall x$ feasible: $f_0(x) \ge L(x, y)$

dual func: $g : \mathbb{R}^m \to \mathbb{R}$ $\quad g(y) := \min_z L(z, y) = \min_z f_0(z) + \sum_{i=1}^m y_i f_i(z)$

then $L(x, y) \ge g(y) \to f_0(x) \ge g(y)$

Dual Problem: $p^* \ge d^* := \max_{y \ge 0} g(y)$ $\quad d^*$: dual optimal value

Nominal Problem: $p^*(u, v) := \min_x f_0(x) : Ax = b + v,\ f_i(x) \le u_i$

Dual: $d^* = \max_{\lambda \ge 0, v} g(\lambda, v)$

Perturbed: $p^*(u, v) := \min_x f_0(x) : Ax = b + v,\ f_i(x) \le u_i$

Dual of Perturbed: $d^* = \max_{\lambda \ge 0, v} g(\lambda, v) - \lambda^T u - v^T v$